Normal frames and linear transports along paths in line bundles. Applications to classical electrodynamics

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Abstract

The definitions and some basic properties of the linear transports along paths in vector bundles and the normal frames for them are recalled. The formalism is specified on line bundles and applied to a geometrical description of the classical electrodynamics. The inertial frames for this theory are discussed.

1. Introduction

The transports along paths in vector bundles [1] are one of the possible generalizations of the parallel transports in these bundles. They are a useful tool for a geometric formulation of quantum mechanics [2]. The frames normal for them are defined as ones in which the transports' matrices are the identity matrix; examples of such frames are the the frames (and possibly coordinates) normal for linear connections on vector bundles [3]. The significance of the normal frames (and coordinates) for the physics is a result of the assertion that they are the mathematical concept representing the physical notion of an 'inertial frame of reference' [3, 4]. From here it follows that the (strong) equivalence principle in gravity physics is a provable theorem [4] and that the scope of its validity can be enlarged to include the gauge theories [3]; in particular, this is valid with respect to classical electrodynamics [5].

The present paper contains a partial review of the general theory of linear transports along paths in vector bundles and the frames normal for them. It is exemplified on 1-dimensional vector bundles, known as line bundles. The formalism is then applied to a geometric description of the classical electrodynamics and the inertial frames for it.

Sections 2 and 3 contain the definitions and some basic properties of the linear transports along paths in vector bundles and of the frames normal for them, respectively. The proofs, extended versions of these results and a lot of details on that items can be found in [1]. Section 4 specifies the results, concerning linear transports and frames normal for them, on line bundles.

In section 5, the results obtained are applied to a geometric description of the classical electromagnetic field. Section 6 is devoted to a brief discussion of the inertial frames for the classical electromagnetic field. Section 7 closes the paper.

2. Linear transports along paths (brief review)

Let (E, π, B) be a complex [1] vector bundle [6, 7] with bundle (total) space E, base B, projection $\pi\colon E \to B$, and homeomorphic fibres $\pi^{-1}(x)$, $x \in B$. [2] The base B is supposed to be a C^1 differentiable manifold. By J and $\gamma\colon J \to B$ are denoted real interval and path in B, respectively. The paths considered are generally *not* supposed to be continuous or differentiable unless their differentiability class is stated explicitly. If γ is a C^1 path, the vector field tangent to it is denoted by $\dot\gamma$.

Definition 2.1. A *linear transport along paths* in the bundle (E, π, B) is a map L assigning to every path γ a map L^γ, *transport along* γ, such that $L^\gamma\colon (s,t) \mapsto L^\gamma_{s\to t}$ where the map

$$L^\gamma_{s\to t}\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s,t \in J, \tag{2.1}$$

called *transport along* γ *from* s *to* t, has the properties:

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \qquad r,s,t \in J, \tag{2.2}$$

$$L^\gamma_{s\to s} = \mathsf{id}_{\pi^{-1}(\gamma(s))}, \qquad s \in J, \tag{2.3}$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v, \qquad \lambda, \mu \in \mathbb{C}, \quad u, v \in \pi^{-1}(\gamma(s)), \tag{2.4}$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Let $\{e_i(s;\gamma)\}$ be a C^1 basis in $\pi^{-1}(\gamma(s))$, $s \in J$. [3] So, along $\gamma\colon J \to B$ we have a set $\{e_i\}$ of bases on $\pi^{-1}(\gamma(J))$ such that the liftings $\gamma \mapsto e_i(\cdot, \gamma)$ of paths are of class C^1.

[1] All of our definitions and results hold also for real vector bundles. Most of them are valid for vector bundles over more general fields too but this is inessential for the following.

[2] When writing $x \in X$, X being a set, we mean "for all x in X" if the point x is not specified (fixed, given) and is considered as an argument or a variable.

[3] Here and henceforth the Latin indices run from 1 to $\dim \pi^{-1}(x)$, $x \in B$. We also assume the Einstein summation rule on indices repeated on different levels.

The *matrix* $\boldsymbol{L}(t,s;\gamma) := \big[L^i{}_j(t,s;\gamma)\big]$ *(along* γ *at* (s,t) *in* $\{e_i\}$*) of a linear transport* L along γ from s to t is defined via the expansion [4] $L^\gamma_{s\to t}\big(e_i(s;\gamma)\big) =: L^j{}_i(t,s;\gamma)e_j(t;\gamma)\ s,t\in J$. A change $\{e_i(s;\gamma)\}\mapsto\{e'_i(s;\gamma) := A^j_i(s;\gamma)e_j(s;\gamma)\}$ via of a non-degenerate matrix $A(s;\gamma) := \big[A^j_i(s;\gamma)\big]$ implies

$$\boldsymbol{L}(t,s;\gamma)\mapsto\boldsymbol{L}'(t,s;\gamma) = A^{-1}(t;\gamma)\boldsymbol{L}(t,s;\gamma)A(s;\gamma) \tag{2.5}$$

or in component form $L'^j{}_i(t,s;\gamma) = \big(A^{-1}(t;\gamma)\big)^j_k L^k{}_l(t,s;\gamma)A^l_i(s;\gamma)$.

Proposition 2.1. *A non-degenerate matrix-valued function* $\boldsymbol{L}\colon (t,s;\gamma)\mapsto\boldsymbol{L}(t,s;\gamma)$ *is a matrix of some linear transport along paths L (in a given field* $\{e_i\}$ *of bases along* γ*) iff*

$$\boldsymbol{L}(t,s;\gamma) = \boldsymbol{F}^{-1}(t;\gamma)\boldsymbol{F}(s;\gamma) \tag{2.6}$$

where $\boldsymbol{F}\colon (t;\gamma)\mapsto\boldsymbol{F}(t;\gamma)$ *is a non-degenerate matrix-valued function.*

Proposition 2.2. *If the matrix* $\boldsymbol{L}$ *of a linear transport* L *along paths has a representation* $\boldsymbol{L}(t,s;\gamma) = {}^\star\boldsymbol{F}^{-1}(t;\gamma)\,{}^\star\boldsymbol{F}(s;\gamma)$ *for some matrix-valued function* ${}^\star\boldsymbol{F}(s;\gamma)$*, then all matrix-valued functions* $\boldsymbol{F}$ *representing* $\boldsymbol{L}$ *via* (2.6) *are given by* $\boldsymbol{F}(s;\gamma) = \boldsymbol{D}^{-1}(\gamma)\,{}^\star\boldsymbol{F}(s;\gamma)$ *where* $\boldsymbol{D}(\gamma)$ *is a non-degenerate matrix depending only on* γ*.*

Let $\{e_i(s;\gamma)\}$ be a smooth field of bases along $\gamma\colon J\to B,\ s\in J$. The explicit local action of the derivation $D\colon\gamma\mapsto D^\gamma\colon s\mapsto D^\gamma_s$, associated to L, on a C^1 lifting of paths λ is

$$D^\gamma_s\lambda = \left[\frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\lambda^j_\gamma(s)\right]e_i(s;\gamma). \tag{2.7}$$

Here the *(2-index) coefficients* $\Gamma^i{}_j$ of the linear transport L are defined by

$$\Gamma^i{}_j(s;\gamma) := \left.\frac{\partial L^i{}_j(s,t;\gamma)}{\partial t}\right|_{t=s} = -\left.\frac{\partial L^i{}_j(s,t;\gamma)}{\partial s}\right|_{t=s} \tag{2.8}$$

and, evidently, uniquely determine the derivation D generated by L. If a matrix $\boldsymbol{F}$ determines the matrix $\boldsymbol{L}$ of a transport L according to proposition 2.1, then

$$\boldsymbol{\Gamma}(s;\gamma) := \big[\Gamma^i{}_j(s;\gamma)\big] = \left.\frac{\partial\boldsymbol{L}(s,t;\gamma)}{\partial t}\right|_{t=s} = \boldsymbol{F}^{-1}(s;\gamma)\frac{\mathrm{d}\boldsymbol{F}(s;\gamma)}{\mathrm{d}s}. \tag{2.9}$$

A change $\{e_i\}\to\{e'_i = A^j_i e_i\}$ of the bases along a path γ with a non-degenerate C^1 matrix-valued function $A(s;\gamma) := \big[A^j_i(s;\gamma)\big]$ implies $\boldsymbol{\Gamma}(s;\gamma) = \big[\Gamma^i{}_j(s;\gamma)\big]\mapsto\boldsymbol{\Gamma}'(s;\gamma) = \big[\Gamma'^i{}_j(s;\gamma)\big]$ with

$$\boldsymbol{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\boldsymbol{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \tag{2.10}$$

3. Normal frames for linear transports (definitions and some results)

Let a linear transport L along paths be given in a vector bundle (E,π,B), $U\subseteq B$ be an arbitrary subset in B, and $\gamma\colon J\to U$ be a path in U.

Definition 3.1. A *frame field (of bases)* in $\pi^{-1}(\gamma(J))$ is called *normal along* γ *for* L if the matrix of L in it is the identity matrix along the given path γ. A *frame field (of bases)* defined on U is called *normal on* U *for* L if it is normal along every path $\gamma\colon J\to U$ in U. The *frame* is called *normal for* L if $U = B$.

[4] Notice the different positions of the arguments s and t in $L^\gamma_{s\to t}$ and in $\boldsymbol{L}(t,s;\gamma)$.

Definition 3.2. A *linear transport along paths* (or *along a path* γ) is called *Euclidean along some (or the given) path* γ if it admits a frame normal along γ. A *linear transport along paths* is called *Euclidean* on U if it admits frame(s) normal on U. It is called *Euclidean* if $U = B$.

Proposition 3.1. *The following statements are equivalent in a given frame* $\{e_i\}$ *over* $U \subseteq B$*:*

(i) *The matrix of* L *is the identity matrix on* U*, i.e.* $\boldsymbol{L}(t, s; \gamma) = \mathbb{1}$ *along every path* γ *in* U*.*

(ii) *The matrix of* L *along every* $\gamma\colon J \to U$ *depends only on* γ*, i.e. it is independent of the points at which it is calculated:* $\boldsymbol{L}(t, s; \gamma) = C(\gamma)$ *where* C *is a matrix-valued function of* γ*.*

(iii) *If* E *is a* C^1 *manifold, the coefficients* ${\Gamma^i}_j(s; \gamma)$ *of* L *vanish on* U*, i.e.* $\boldsymbol{\Gamma}(s; \gamma) = 0$ *along every path* γ *in* U*.*

(iv) *The explicit local action of the derivation* D *along paths generated by* L *reduces on* U *to differentiation of the components of the liftings with respect to the path's parameter if the path lies entirely in* U*:* $D^\gamma_s \lambda = \frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} e_i(s; \gamma)$ *where* $\lambda = \lambda^i e_i$ *is a* C^1 *lifting of paths and* $\lambda\colon \gamma \mapsto \lambda_\gamma$*.*

(v) *The transport* L *leaves the vectors' components unchanged along any path in* U*, viz. we have* $L^\gamma_{s\to t}\big(u^i e_i(s; \gamma)\big) = u^i e_i(t; \gamma)$ *for all* $u^i \in \mathbb{C}$*.*

(vi) *The basic vector fields are* L*-transported along any path* $\gamma\colon J \to U$*:* $L^\gamma_{s\to t}\big(e_i(s; \gamma)\big) = e_i(t; \gamma)$*.*

Remark 3.1. It is valid the equivalence $\boldsymbol{L}(t, s; \gamma) = \mathbb{1} \iff \boldsymbol{F}(s; \gamma) = \boldsymbol{B}(\gamma)$ with $\boldsymbol{B}$ being a matrix-valued function of the path γ only. According to proposition 2.2, this dependence is inessential and, consequently, in a normal frame, we can always choose representation (2.6) with $\boldsymbol{F}(s; \gamma) = \mathbb{1}$.

Corollary 3.1. *Every linear transport along paths is Euclidean along every fixed path without self-intersections.*

Theorem 3.1. *A linear transport along paths admits frames normal on some set (resp. along a given path) if and only if its action along every path in this set (resp. along the given path) depends only on the initial and final point of the transportation but not on the particular path connecting these points. In other words, a transport is Euclidean on* $U \subseteq B$ *iff it is path-independent on* U*.*

Proposition 3.2. *Let* L *be a linear transport along paths in* (E, π, M)*,* E *and* M *being* C^1 *manifolds, and* L *be Euclidean on* $U \subseteq M$ *(resp. along a* C^1 *path* $\gamma\colon J \to M$*). Then the matrix* $\boldsymbol{\Gamma}$ *of its coefficients has the representation*

$$\boldsymbol{\Gamma}(s; \gamma) = \sum_{\mu=1}^{\dim M} \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \equiv \Gamma_\mu(\gamma(s))\dot\gamma^\mu(s) \tag{3.2}$$

in any frame $\{e_i\}$ *along every (resp. the given)* C^1 *path* $\gamma\colon J \to U$*, where* $\Gamma_\mu = \big[{\Gamma^i}_{j\mu}\big]_{i,j=1}^{\dim \pi^{-1}(x)}$ *are some matrix-valued functions, defined on an open set* V *containing* U *(resp.* $\gamma(J)$*) or equal to it, and* $\dot\gamma^\mu$ *are the components of* $\dot\gamma$ *in some frame* $\{E_\mu\}$ *along* γ *in the bundle space tangent to* M*,* $\dot\gamma = \dot\gamma^\mu E_\mu$*. The functions* ${\Gamma^i}_{j\mu}$ *are termed* 3-index coefficients *of* L*.*

Let U be an open set, e.g. $U = M$. If we change the frame $\{E_\mu\}$ in the bundle space tangent to M, $\{E_\mu\} \mapsto \{E'_\mu = B^\nu_\mu E_\nu\}$ with $B = \big[B^\nu_\mu\big]$ being non-degenerate matrix-valued function, and simultaneously the bases in the fibres $\pi^{-1}(x)$, $x \in M$, $\{e_i|_x\} \mapsto \{e'_i|_x = A^j_i(x) e_j|_x\}$, then, from (2.10) and (3.2), we see that Γ_μ transforms into Γ'_μ such that

$$\Gamma'_\mu = B^\nu_\mu A^{-1}\Gamma_\nu A + A^{-1}E'_\mu(A) = B^\nu_\mu A^{-1}\big(\Gamma_\nu A + E_\nu(A)\big) \tag{3.3}$$

where $A := \big[A^j_i\big]_{i,j=1}^{\dim \pi^{-1}(x)}$ is non-degenerate and of class C^1.

Theorem 3.2. *A* C^2 *linear transport* L *along paths is Euclidean on a neighborhood* $U \subseteq M$ *if and only if in every frame the matrix* $\boldsymbol{\Gamma}$ *of its coefficients has a representation* (3.2) *along every* C^1 *path* γ *in* U *in which the matrix-valued functions* Γ_μ*, defined on an open set containing* U *or equal to it, satisfy the equalities*

where $x \in U$ and

$$R_{\mu\nu}(-\Gamma_1,\dots,-\Gamma_{\dim M}) := -\frac{\partial \Gamma_\mu}{\partial x^\nu} + \frac{\partial \Gamma_\nu}{\partial x^\mu} + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu. \tag{3.5}$$

in a coordinate frame $\{E_\mu = \frac{\partial}{\partial x^\mu}\}$ in a neighborhood of x

Theorem 3.3. *A linear transport L along paths is Euclidean on a submanifold N of M if and only if in every frame $\{e_i\}$, in the bundle space over N, the matrix of its coefficients has a representation (3.2) along every C^1 path in N and, for every $p_0 \in N$ and a chart (V,x) of M such that $V \ni p_0$ and $x(p) = (x^1(p),\dots,x^{\dim N}(p), t_0^{\dim N+1},\dots,t_0^{\dim M})$ for every $p \in N\cap V$ and constant numbers $t_0^{\dim N+1},\dots,t_0^{\dim M}$, the equalities*

$$\big(R^N_{\alpha\beta}(-\Gamma_1,\dots,-\Gamma_{\dim N})\big)(p) = 0, \qquad \alpha,\beta = 1,\dots,\dim N \tag{3.6}$$

hold for all $p \in N\cap V$ and

$$R^N_{\alpha\beta}(-\Gamma_1,\dots,-\Gamma_{\dim N}) := R_{\alpha\beta}(-\Gamma_1,\dots,-\Gamma_{\dim M}) = -\frac{\partial \Gamma_\alpha}{\partial x^\beta} - \frac{\partial \Gamma_\beta}{\partial x^\alpha} + \Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha. \tag{3.7}$$

Here $\Gamma_1,\dots,\Gamma_{\dim N}$ are first $\dim N$ of the matrices of the 3-index coefficients of L in the coordinate frame $\{\frac{\partial}{\partial x^\mu}\}$ in the tangent bundle space over $N\cap V$.

4. Linear transports and normal frames in line bundles

Let (E,π,M) be one-dimensional vector bundle over a C^1 manifold M; such bundles are called line bundles. Thus the (typical) fibre of (E,π,M) can be identified with $\mathbb{C}$ (resp. $\mathbb{R}$ in the real case) and then the fibre $\pi^{-1}(x)$ over $x \in M$ will be an isomorphic image of $\mathbb{C}$ (resp. $\mathbb{R}$ in the real case). Let $\gamma\colon J \to M$ be of class C^1 and L be a linear transport along paths in (E,π,M). A frame $\{e\}$ along γ consists of a single non-zero vector field $e\colon (s;\gamma) \to e(s;\gamma) \in \pi^{-1}(\gamma(s))\backslash\{0\}$, $s \in J$, and in it the matrix of L^γ at $(t,s) \in J\times J$ is simply a number $\boldsymbol{L}(t,s;\gamma) \in \mathbb{C}$, $L^\gamma_{s\to t}(ue(s;\gamma)) = u\boldsymbol{L}(t,s;\gamma)e(t;\gamma)$ for $u \in \mathbb{C}$ and $s,t \in J$. By proposition 2.1, the general form of $\boldsymbol{L}$ is

$$\boldsymbol{L}(t,s;\gamma) = \frac{f(s;\gamma)}{f(t;\gamma)} \tag{4.1}$$

where $f\colon (s;\gamma) \mapsto f(s;\gamma) \in \mathbb{C}\backslash\{0\}$ is defined up to (left) multiplication with a function of γ (proposition 2.2). Respectively, due to (2.9), the matrix of the coefficient(s) of L is

$$\boldsymbol{\Gamma}(s;\gamma) = \frac{\partial \boldsymbol{L}(t,s;\gamma)}{\partial s}\Big|_{t=s} = \frac{1}{f(s;\gamma)}\frac{\mathrm{d}f(s;\gamma)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\big[\ln(f(s;\gamma)\big] \tag{4.2}$$

and [1, equation (2.31)] takes the form

$$\boldsymbol{L}(t,s;\gamma) = \exp\Big(-\int_s^t \boldsymbol{\Gamma}(\sigma;\gamma)\,\mathrm{d}\sigma\Big). \tag{4.3}$$

A change $e(s;\gamma) \mapsto e'(s;\gamma) = a(s;\gamma)e(s;\gamma)$, with $a(s;\gamma) \in \mathbb{C}\backslash\{0\}$, of the frame $\{e\}$ implies (see (2.5) and (2.10))

$$\boldsymbol{L}(t,s;\gamma) \mapsto \boldsymbol{L}'(t,s;\gamma) = \frac{a(s;\gamma)}{a(t;\gamma)}\boldsymbol{L}(t,s;\gamma) \tag{4.4a}$$

$$\boldsymbol{\Gamma}(s;\gamma) \mapsto \boldsymbol{\Gamma}'(s;\gamma) = \boldsymbol{\Gamma}(s;\gamma) + \frac{\mathrm{d}}{\mathrm{d}s}\big[\ln(a(s;\gamma)\big]. \tag{4.4b}$$

The explicit local action of the derivation D along paths generated by L is

where $\lambda \in \mathrm{PLift}^1(E,\pi,M)$ and (2.7) was used.

Let us now look on the normal frames on one-dimensional vector bundles.

A frame $\{e\}$ is normal for L along γ (resp. on U) iff in that frame equation (4.1) holds with

$$f(s;\gamma) = f_0(\gamma) \tag{4.6}$$

where $\gamma\colon J \to M$ (resp. $\gamma\colon J \to U$) and $f_0\colon \gamma \mapsto f_0(\gamma) \in \mathbb{C}\backslash\{0\}$ (see remark 3.1 and proposition 3.1). Since, in a frame normal along γ (resp. on U), it is fulfilled

$$\boldsymbol{L}(t,s;\gamma) = \mathbb{1}, \quad \boldsymbol{\Gamma}(s;\gamma) = 0 \tag{4.7}$$

for the given path γ (resp. every path in U), in every frame $\{e' = ae\}$, we have

$$\boldsymbol{L}'(t,s;\gamma) = \frac{a(s;\gamma)}{a(t;\gamma)}, \quad \boldsymbol{\Gamma}'(s;\gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\big[\ln(a(s;\gamma)\big]. \tag{4.8}$$

In addition, for Euclidean on $U \subseteq M$ transport L, the representation

$$\boldsymbol{\Gamma}'(s;\gamma) = \Gamma'_\mu(\gamma(s))\dot{\gamma}'^{\,\mu}(s) \tag{4.9}$$

holds for every C^1 path $\gamma\colon J \to U$ and some $\Gamma'_\mu\colon V \to \mathbb{C}$ with V being an open set such that $V \supseteq U$ (proposition 3.2). This means (see theorem 3.1 and [1, theorem 4.2]) that (4.8) holds for

$$a(s;\gamma) = a_0(\gamma(s)), \tag{4.10}$$

where $a_0\colon U \to \mathbb{C}\backslash\{0\}$, and, consequently, the equality (4.9) can be satisfied if we choose

$$\Gamma'_\mu = E_\mu(a) \tag{4.11}$$

with $a\colon V \to \mathbb{C}$, $a|_U = a_0$ and $\{E_\mu\}$ being a frame in the bundle space tangent to M which, in particular, can be a coordinate one, $E_\mu = \frac{\partial}{\partial x^\mu}$. Of course, if U is not an open set, this choice of Γ'_μ is not necessary; for example, the equality (4.9) will be preserved, if to the r.h.s. of (4.11) is added a function G'_μ such that $G'_\mu\dot{\gamma}'^{\,\mu} = 0$.

By virtue of (3.3), the functions Γ_μ and Γ'_μ in two arbitrary pairs of frames $(\{e\},\{E_\mu\})$ and $(\{e' = ae\},\{E'_\mu = B^\nu_\mu E_\nu\})$, respectively, are connected via

$$\Gamma'_\mu = B^\nu_\mu\Gamma_\nu + \frac{1}{a}E'_\mu(a) = B^\nu_\mu\big(\Gamma_\nu + E_\nu(\ln a)\big) \tag{4.12}$$

and, consequently, with respect to changes of the frames in the tangent bundle space over M, when $a = 1$, they behave like the components of a covariant vector field (one-form). Therefore, on an open set U, e.g. $U = M$, the quantity

$$\omega = \Gamma_\mu E^\mu, \tag{4.13}$$

where $\{E^\mu\}$ is the coframe dual to $\{E_\mu\}$ (in local coordinates: $E_\mu = \frac{\partial}{\partial s^\mu}$ and $E^\mu = \mathrm{d}x^\mu$), is a 1-form over M (with respect to changes of the local coordinates on M or of the frames in the (co)tangent bundle space over M). However, it depends on the choice of the frame $\{e\}$ in the bundle space E and a change $e \mapsto e' = ae$ implies

$$\omega \mapsto \omega' = \omega + (E_\nu(\ln a))E^\nu = \omega + (E'_\nu(\ln a))E'^{\,\nu}. \tag{4.14}$$

Using the 1-form (4.13), we see that

$$\boldsymbol{\Gamma}(s;\gamma) = \omega|_{\gamma(s)}(\dot{\gamma}(s)) \tag{4.15}$$

and (4.3) can be rewritten as

$$\boldsymbol{L}(t,s;\gamma) = \exp\Big(-\int^{\gamma(t)}\omega\Big) \tag{4.16}$$

where the integration is along some path in U (on which the transport L is Euclidean). Hence $\boldsymbol{L}$ (or L) depends only on the points $\gamma(s)$ and $\gamma(t)$, not on the particular path connecting them, as it should be (theorem 3.1). The self-consistency of our results is confirmed by the equation

$$R_{\mu\nu}|_U = 0 \tag{4.17}$$

which is a consequence of (4.11) and (3.5) and which is a necessary and sufficient condition for the existence of frames normal on an open set U (theorem 3.2).

We end this section with a remark that frames normal along injective paths always exist (corollary 3.1), but on an arbitrary submanifold $N \subseteq M$ they exist iff the functions Γ_μ satisfy the conditions (3.6) with $x \in N$ in the coordinates described in theorem 3.3.

5. Bundle description of the classical electromagnetic field

Now we would like to apply the above formalism to a description of the classical electromagnetic field. Before going on, we should say that the accepted natural formalism in gauge field theories, in particular in the electrodynamics, is via connections on vector bundles [8,9]. This approach deserves a special investigation and we shall return to it in a separate paper (see [3]). Below we sketch an equivalent technique for an electromagnetic field.

Recall [9, 10], the classical electromagnetic field is described via a real 1-form A over a 4-dimensional real manifold M (endowed with a (pseudo-)Riemannian metric g and) representing the space-time model and, usually, identified with the Minkowski space M^4 of special relativity or the (pseudo-)Riemannian space V_4 of general relativity. [5] The electromagnetic field itself is represented by the two-form $F = \mathrm{d}A$, where "d" denotes the exterior derivative operator, with local components (in some local coordinates $\{x^\mu\}$)

$$F_{\mu\nu} = -\frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial A_\nu}{\partial x^\mu}. \tag{5.1}$$

As it is well known, the electromagnetic field, the Maxwell equations describing it, and its (minimal) interactions with other objects are invariant under a gauge transformation

$$A_\mu \mapsto A'_\mu = A_\mu + \frac{\partial \lambda}{\partial x^\mu} \tag{5.2}$$

or $A \mapsto A' = A + \mathrm{d}\lambda$, where λ is a C^2 function. As is almost evident, the electromagnetic field is invariant under simultaneous changes of the local coordinate frame, $E_\mu = \frac{\partial}{\partial x^\mu} \mapsto E'_\mu = B^\nu_\mu E_\nu$ with $B^\nu_\mu := \frac{\partial x^\nu}{\partial x'^\mu}$, and a gauge transformation (5.2):

$$A_\mu \mapsto A'_\mu = B^\nu_\mu A_\nu + E'_\mu(\lambda) = B^\nu_\mu \Big(A_\nu + \frac{\partial \lambda}{\partial x^\nu} \Big). \tag{5.3}$$

A simple calculation shows that under the transformation (5.3), the quantities (5.1) transform like components of an antisymmetric tensor,

$$F_{\mu\nu} \mapsto F'_{\mu\nu} = B^\sigma_\mu B^\tau_\nu F_{\sigma\tau} \tag{5.4}$$

due to which the 2-form F remains unchanged, $F = \mathrm{d}A = \mathrm{d}A'$. Notice, above A'_μ are *not* the components of A in $\{E'_\mu\}$ unless $\lambda = \mathrm{const}$ while $F'_{\mu\nu}$ *are* the components of F in $\{E'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu\}$.

The similarity between (5.3) and (4.12) is obvious and implies the idea of identifying (on an open set, neighborhood) the electromagnetic potentials A_μ with the matrices (functions, in the particular case) Γ_μ of the 3-index coefficients of some linear transport along paths in a 1-dimensional vector bundle (E, π, M). This can be done as follows.

Let M be a real 4-dimensional manifold, representing the space-time model, and (E, π, M) be a 1-dimensional real vector bundle over it. [6] We *identify the potentials A_μ of an electromagnetic field*

[5] The particular choice of M is insignificant for the following.

[6] The consideration of the real case does not change the above results with the exception that $\mathbb{C}$ should be replaced

with the (local) coefficients of a linear transport L along paths in (E, π, M) whose matrix has the representation (4.9) *(along every path and in every pair of frames).* Hence, the 3-index coefficients of L are uniquely defined and supposed to be (arbitrarily) fixed in some pair of frames.

Since the 3-index coefficients of linear transport are defined in a pair of frames $(\{e\}, \{E_\mu\})$, $\{e\}$ in the bundle space E and $\{E_\mu\}$ in the tangent bundle space $T(M)$, the change (5.3) expresses simply the transformation of A_μ under the pair of changes $e \mapsto e' = ae$ and $E_\mu \mapsto E'_\mu = B^\nu_\mu E_\nu$ and is a consequence of (4.12) if we put

$$a = \mathrm{e}^\lambda. \tag{5.5}$$

It should be emphasized, now the (pure) gauge transformation (5.2) appears as a special case of (5.3), corresponding to a change of the frame in E and a fixed frame in $T(M)$. [7] This means that, in the approach proposed, the change (5.2) is directly incorporated in the definition of the field potential A. This conclusion is in contrast to the situation in classical electrodynamics as there the change (5.2) is a simple observation of 'additional' invariance of the field, which is not connected with the geometrical interpretation of the theory.

Defining the electromagnetic field (strength) by $F = \mathrm{d}A$, the equality (5.1) remains valid in a coordinate frame $\{E_\mu = \partial/\partial x^\mu\}$. Since A and F possess all of the properties they must have in classical electrodynamics, they represent an equivalent description of electromagnetic field. The only difference with respect to the classical description is the clear geometrical meaning of these quantities, as a consequence of which an electromagnetic field can be identified with a linear transport along paths in a one-dimensional vector bundle over the space-time. With a little effort, one can show that the proposed treatment of electromagnetic field is equivalent to the modern one in the bundle picture of gauge theories (see, e.g., [8] or [12]), where the electromagnetic potentials are regarded as coefficients of a suitable linear connection.

In the approach proposed, the different gauge conditions, which are frequently used, find a natural interpretation as a partial fix of the class of frames in the bundle space employed. For instance, any one of the gauges in the table 5.1 on this page corresponds to a class of frames for which (5.3) holds for $B^\nu_\mu = \delta^\nu_\mu$, δ^ν_μ being the Kronecker deltas, and λ subjected to a condition given in the table. [8]

Table 5.1: Examples of gauge conditions

Gauge	Condition on A	Condition on λ	Condition on φ
Lorenz [a]	$\partial^\mu A_\mu = 0$	$\partial^\mu \partial_\mu \lambda = 0$	$\partial^\mu \partial_\mu \varphi = -\partial^\mu \partial_\mu \lambda$
Coulomb [b]	$\partial^k A_k = 0$	$\partial^k \partial_k \lambda = 0$	$\partial^k \partial_k \varphi = -\partial^k \partial_k \lambda$
Hamilton	$A_0 = 0$	$\lambda(x) = \lambda(x^1, x^2, x^3)$	$\varphi(x) = \varphi(x^1, x^2, x^3)$
Axial	$A_3 = 0$	$\lambda(x) = \lambda(x^0, x^1, x^2)$	$\varphi(x) = \varphi(x^0, x^1, x^2)$

[a] The Lorenz condition and gauge are named in honor of the Danish theoretical physicist Ludwig Valentin Lorenz (1829–1891), who has first published it in 1867 [13] (see also [14, pp. 268-269, 291]); however this condition was first introduced in lectures by Bernhard G. W. Riemann in 1861 as pointed in [14, p. 291]. It should be noted that the *Lorenz* condition/gauge is quite often erroneously referred to as the Loren*tz* condition/gauge after the name of the Dutch theoretical physicist Hendrik Antoon Lorentz (1853–1928) as, e.g., in [15, p. 18] and in [16, p. 45].

[b] In this row the summation over k is from 1 to 3.

In the table 5.1 on the current page φ is a C^1 function describing the arbitrariness in the choice of λ, i.e. if a gauge condition is valid for λ, then it holds also for $\lambda + \varphi$ instead of λ.

[7] Cf. a similar conclusion in [11, p. 178], in which a gauge transformation, in a general gauge theory, is interpreted as a change in fibre coordinates of a principle bundle.

[8] Below M is supposed to be endowed with a (pseudo-)Riemannian metric $g_{\mu\nu}$, the coordinates to be numbered as x^0, x^1, x^2, and x^3, x^0 to be the 'time' coordinate, $\partial_\mu := \partial/\partial x^\mu$, and $\partial^\mu := g^{\mu\nu}\partial_\nu$ with $[g^{\mu\nu}] := [g_{\mu\nu}]^{-1}$.

6. Normal and inertial frames

Comparing (5.1) with (3.5), we get [9]

$$F_{\mu\nu} = R_{\mu\nu}(-A_0, -A_1, -A_2, -A_3). \tag{6.1}$$

Thus, the electromagnetic field tensor F is completely responsible for the existence of frames normal for L (theorems 3.2 and 3.3). For example, if U is an open set, frames normal on $U \subseteq M$ for L exist iff $F|_U = 0$, i.e. if electromagnetic field is missing on U. [10] Also, if N is a submanifold of M, frames normal on U for L exist iff in the special coordinates $\{x^\mu\}$, described in theorem 3.3, is valid $F_{\alpha\beta}|_U = 0$ for $\alpha, \beta = 1, \dots, \dim N$. In the context of theorem 3.1, we can say that an electromagnetic field admits frames normal on $U \subseteq M$ iff the linear transport L corresponding to it is path-independent on U (along paths lying entirely in U). Thus, if L is path-dependent on U, the field does not admit frames normal on U. This important result is the classical analogue of a quantum effect, know as the Aharonov-Bohm effect [18, 19], whose essence is that the electromagnetic potentials directly, not only through the field tensor F, can give rise to observable physical results.

Let us now turn our attention to the physical meaning of the normal frames corresponding to a given electromagnetic field which is described, as pointed above, via a linear transport L along paths in a line vector bundle over the space-time M.

Suppose L is Euclidean on a neighborhood $U \subseteq M$. As a consequence of (6.1) and theorem 3.2, we have $F|_U = \mathrm{d}A|_U = 0$, i.e. on U the electromagnetic field strength vanishes and hence the field is a pure gauge on U,

$$A_\mu|_U = \frac{\partial f_0}{\partial x^\mu}\Big|_U \tag{6.2}$$

for some C^1 function f_0 defined on an open set containing U or equal to it. As we know from proposition 3.1, in a frame $\{e'\}$ normal on U for L vanish the 2-index coefficients of L along any path γ in U:

$$\mathbf{\Gamma}'(s;\gamma) = A'_\mu(\gamma(s))\dot\gamma^\mu(s) = 0 \tag{6.3}$$

for every $\gamma\colon J \to U$ and $s \in J$. Using (6.2), it is trivial to see that any transformation (5.3) with

$$\lambda = -f_0 \tag{6.4}$$

transforms A_μ into A'_μ such that

$$A'_\mu|_U = 0 \tag{6.5}$$

(irrespectively of the frames $\{E_\mu\}$ and $\{E'_\mu\}$ in the tangent bundle over M). Hence, by (6.3) the one-vector frame $\{e' = \mathrm{e}^{-f_0}e\}$ in the bundle space E is normal for L on U. Therefore, in the frame $\{e'\}$, vanish not only the 2-index coefficients of L but also its *3-index* ones, i.e. $\{e'\}$ is a frame *strong normal* on U for L. Applying (5.3) one can verify, *all frames strong normal on a neighborhood U for L are obtainable from $\{e'\}$ by multiplying its vector e' by a function f such that $\frac{\partial f}{\partial x^\mu}\big|_U = 0$*, i.e. they are $\{b\mathrm{e}^{-f_0}e\}$ with $b \in \mathbb{R}\backslash\{0\}$ as U is a neighborhood. Thus, every frame normal on a neighborhood U for L is strong normal on U for L and vice versa.

A frame (of reference) in the bundle space in which (6.5) holds on a subset $U \subseteq M$, will be called *inertial on U for the electromagnetic field* considered. In other words, the frames inertial on U for a given electromagnetic field are the ones in which its potentials vanish on U. Thus, every frame inertial on U is strong normal on it and vice versa.

So, in a frame inertial on $U \subseteq M$ for an electromagnetic field it is not only a pure gauge, but in such a frame its potentials vanish on U. Relying on the results obtained (see [20–22]), we can assert the existence of frames inertial at a single point and/or along paths without self-intersections for every electromagnetic field, while on submanifolds of dimension not less than two such frames exist only as

[9] In this section, we assume the Greek indices to run over the range 0, 1, 2, 3.

[10] Elsewhere we shall prove that the components $F_{\mu\nu}$ completely describe the curvature of L which agrees with the interpretation of $F_{\mu\nu}$ as components of the curvature of a connection on a vector bundle in the gauge theories [8, 9, 17].

an exception if (and only if) some additional conditions are satisfied, i.e. for some particular types of electromagnetic fields.

Now we would like to make a link with the paper [4] in which was demonstrated that the (ordinary strong) equivalence principle is a provable theorem and the inertial frames in a gravity theory based on a linear connection (or other derivation) are the frames normal for it. For reasons given a few lines below, such frames will be called *inertial for the gravitational field* under consideration.

Let there be given a physical system consisting of pure or, possibly, interacting gravitational and electromagnetic fields which are described via, respectively, a linear connection ∇ in the tangent bundle $(T(M), \pi_T, M)$ (or the tensor algebra) over the space-time M and a linear transport along paths in a 1-dimensional vector bundle (E, π_E, M) over M. On one hand, as we saw above, the frames inertial for an electromagnetic field, if any, in the bundle space E are completely independent of any frame in the bundle space $T(M)$ tangent to M. On the other hand, the frames inertial for the gravity field, i.e. the ones normal for ∇, if any, are frames in $T(M)$ and have nothing in common with the frames in E, in particular with the frames normal for L, if any. Consequently, if there is a frame $\{E_\mu\}$ in $T(M)$ inertial on $U \subseteq M$ for the gravity field and a frame $\{e\}$ in E inertial on the same set U for the electromagnetic field, the frame $\{e \times E_\mu\} = \{(e, E_\mu)\}$ in the bundle space of the bundle $(E \times T(M), \pi_E \times \pi_T, M \times M)$ over $M \times M$ can be called simply *inertial* on U (for the system of gravity and electromagnetic fields). [11] Thus, in an inertial frame, if any, the potentials of both, gravity and electromagnetic, fields vanish. Relying on the results obtained in this work, as well as on the ones in [4, 20–22], we can assert the existence of inertial frames at every single space-time point and/or along every path without self-intersections in it. On submanifolds of dimension higher than one, inertial frames exist only for some exceptional configurations of the fields which can be described on the base of the results in the cited works.

7. Conclusion

One of the purposes of the present paper was to exemplify the general theory of linear transports along paths and the frames normal for them on line bundles. The application of the so-obtained results to the classical electrodynamics gives rise to a geometric interpretation of the electromagnetic field as a linear transport in a line bundle and to an introduction of inertial frames for this field. As pointed in [5], the linear transport, describing the electromagnetic field in our approach, is in fact the parallel transport generated by the linear connection describing it in the well known its geometrical interpretation [16].

The coincidence of the normal and inertial frames for the electromagnetic field expresses the equivalence principle for that field [5]. Generally this principle is a provable theorem and it is always valid at any single point of along given path (without selfintersections) as these are the only cases when normal frames for a linear connection or transport always exist.

For a free electromagnetic field, the line bundle mentioned above remains unspecified. However, if an interaction of that field with other one is presented, the line bundle under question can be identified or uniquely connected with a bundle (over the spacetime) whose sections represent the latter field. Moreover, in such a situation the equivalence principle can be used to justify the so-called minimal coupling (principle).

The considerations in this work confirm our opinion that the frames (and possibly coordinates) in bundle spaces, in which some physical fields 'live', should be regarded as parts of the frames of references with respect to which a physical system is investigated. [12] In the particular case of an electromagnetic field, these are the one-vector field frames $\{e\}$ in the bundle space E of the line bundle (E, π, M) in which the field is describe via a linear transport L along paths. With respect

[11] For purposes which will be explained elsewhere, the product bundle $(E \times T(M), \pi_E \times \pi_T, M \times M)$ is better to be replaced with the bundle (F, π, M), with $\mathsf{F} := \{(\xi, \eta) \in E \times T(M) | \pi_E(\xi) = \pi_T(\eta)\}$ and $\pi(\xi, \eta) := \pi_E(\xi) = \pi_T(\eta) \in M$ for $(\xi, \eta) \in \mathsf{F}$, i.e. $\pi^{-1}(x) = \pi_E^{-1}(x) \times \pi_T^{-1}(x)$, $x \in M$. Evidently, (F, π, M) is isomorphic to the Whitney sum [6, sect. 1.29] of (E, π_E, M) and $(T(M), \pi_T, M)$ and the standard fibre of (F, π, M) can be identified with $\mathbb{R} \times \mathbb{R}^4 = \mathbb{R}^5$ as (E, π_E, M) is 1-dimensional and $\dim M = 4$.

to $\{e\}$ is defined the sole coefficient of L and with respect to a pair $(\{e\},\{E_\mu\})$, with $\{E_\mu\}$ being a frame in the bundle tangent to the spacetime, are defined the (3-index) coefficients of L which, by definition, coincide with the components A_μ of the 4-vector potential of the electromagnetic field. Since A_μ are observable (if one beliefs in the Aharonov-Bohm effect) and $\{E_\mu\}$, usually constructed from some local coordinates x^μ ($E_\mu = \frac{\partial}{\partial x^\mu}$), is an essential path of the frames of reference, one can conclude that $\{e\}$ should be a part of the frame of reference and there should exist a method of its experimental/ laboratory realization.

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